Discrete-element computation of response functions in static rectangular assemblies of polygonal particles

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Abstract

The averaged stress and strain response functions of granular aggregates are investigated numerically. We use the discrete-element method (DEM) to generate granular packings consisting of soft convex polygonal particles, i.e., the simulation geometry is two-dimensional. Packings are prepared in a rectangular container. To determine the stress response of a packing, we apply an external load to a single grain from the top layer of the assembly, with a force small enough not to cause structural rearrangements. Measuring the average vertical normal stress response at different heights of the sample, we find that the shape of the stress response function depends on the regularity of the granular assembly**.** For packings with strong spatial order, the average stress response shows a behaviour corresponding to that of hyperbolic continuum equations. As the amount of contact disorder increases, there is no wave-like stress propagation anymore and a behaviour emerges that would rather be predicted by elliptic equations. Furthermore, we show that not only *geometric* disorder but also large values of static friction coefficients, which may be linked to *force* disorder, lead to elliptic equations. Finally, we determine the strain response for a rectangular sample that consists of monodisperse particles.

Keywords: discrete element simulation; granular matter; stress response; strain response

1 Introduction

During the last few years, extensive research has been devoted to the study of the mechanical properties of granular materials. To some extent, this is due to their importance in applications within various industrial branches such as the pharmaceutical, agricultural, geotechnical, and energy production industries. On the other hand, granular media are also interesting from a fundamental point of view – so far no generally applicable coherent theoretical description for their macro-states is available, which calls for more scientific activity.

In particular, the stress distribution under a sand pile has attracted much attention leading to theoretical [1-3], experimental [4-5] and numerical studies [6-7]. Here we will consider a somewhat simpler arrangement, *viz.* a rectangular sample of granulate that we perturb applying an external force. In naturally occuring systems but also in industrial applications, a granular aggregate usually is not in an ordered state. Therefore, the question arises to what extent disorder affects the response of the material. Determining stress and strain response functions resulting from localized perturbations has, for various reasons, been one of the more interesting tasks of researchers on granular materials in the physics community, both experimentally [8-11] and theoretically [12-14]. One motivation is that conclusions on the nature of macroscopic continuum equations describing stress propagation in a granular material can be drawn from these response functions.

The stress response (to a point force) under an assembly of grains exhibits puzzling properties. In some cases, it shows wave-like propagation underneath the point where the force is applied and in others the response is elastic. What is observed depends strongly on the packing structure of the granular assembly. For packings with strong spatial order, a stress double peak may appear below the point where the force is applied, which seems explicable in terms of wave-like stress propagation describable by hyperbolic continuum equations [1, 3,13,15], whereas when the amount of disorder increases, meaning that the packing has large contact disorder, then there is a single peak, hinting at an elastic-like response, describable by elliptic continuum equations. In the theory of elasticity, the shape of the stress response consists of a single peak. The width of the response increases with the distance from the point of application of the external load.

Both cases have been observed in experimental models set up by Junfei Geng et al. [8] using photo-elastic polymer material. Hence, two rectangular systems consisting of the same material may have different stress responses to a point force, depending on the way they were prepared.

Moreover, the coefficient of static friction plays an important role in the stress response. When this coefficient is very small, the response to an external load may have a double-peaked shape. On the other hand, when it is large, the double peak may be present but much less pronounced, as has been observed experimentally [10] for rectangular packings with different friction coefficients.

We focus on investigating numerically the mechanical properties of a static granular assembly, determining the stress response for a rectangular system of granular material with different amounts of disorder. The particles are initialized on regular triangular or rectangular lattices and allowed to rearrange under the influence of gravity. We consider different frictional properties and different values of the applied vertical force.

This paper is organised as follows. In section 2, we describe details of the simulation setup. In section 3, we determine the angular distribution of the contact forces for different packings. We then present simulation results on the stress response of two-dimensional rectangular aggregates of particles for various packing orders in section 4. The stress response for different values of static friction and different values of the applied external load is discussed in section 5. Next, section 6 is devoted to the calculation of the stress response for polydisperse systems with smooth and with rough bottoms. Comparison is made with existing experimental results. In section 7, we perform a quantitative comparison between simulation data for the stress response and an isotropic elasticity prediction. The calculation of the macroscopically averaged strain response function inside a granular aggregate is presented in section 8. Section 9 summarizes our results.

2 Simulation geometry

In order to complement existing experimental data and to investigate the stress response for granular materials where particles were placed on either a smooth bottom or a rough bottom and to compare numerical data with analytical results, and finally to obtain the strain response, we have performed extensive numerical simulations of various grain packings by layer-wise deposition of particles. To study packings of varying degrees of spatial order, we simulated four different types of samples by constructing monodisperse, bidisperse, and polydisperse assemblies of roundish particles (these were 16-sided regular polygons) as well as packings of regular pentagons. For the monodisperse distribution, we used particles with a fixed diameter of 0.9 cm, the only disorder being variations in local angular orientation after relaxation of the sample. In bidisperse systems, we used mixtures of two round particle types with diameters of 0.9 cm and 0.7 cm, respectively. The pentagons had the an edge length of 0.9 cm and for the polydisperse distribution, the radius of the particles was distributed uniformly between 0.36 cm and 0.54 cm.

Simulations were carried out for two dimensional systems via the "discrete element method (DEM)" originally developed by Cundall and Strack [16] for the simulation of particulate materials. The basic structure of DEM consists of a simple loop that is composed of three steps: collision detection, force calculation and time integration. We use soft convex particles and calculate forces from the particle overlap [17].

In order to determine the position and orientation of each particle, we solve Newton's and Euler's equations of motion (1), using an explicit algorithm, a fifth-order Gear predictor-corrector method [18], usually with a fixed time step:

$$
\begin{aligned}
m_i \ddot{\mathbf{r}}_i &= \mathbf{F}_i + \sum_{j=1}^{n} \mathbf{F}_{ij} \\
I_i \ddot{\phi}_i &= T_i + \sum_{j=1}^{n} T_{ij} .
\end{aligned} \tag{1}
$$

Here, m_i and $\ddot{r}_i$ denote the mass and linear acceleration of particle $\boldsymbol{i}$. $\mathbf{F}_i$ is the external force acting on particle i; in our problem, gravity is the only external force. $\mathbf{F}_{ij}$ is the force produced by the particle touching particle $\boldsymbol{i}$ in contact j. T_{ij} is the torque generated by this force. I_i and $\ddot{\phi}_i$ are the moment of inertia and angular acceleration of particle $\boldsymbol{i}$ about its centre of mass, respectively. Time is advanced after the initialization procedure of the assembly until the latter has relaxed to a steady state. Then the force perturbation is applied and response functions are measured after the system has relaxed to a steady state again.

Since we consider non-cohesive particles in our simulation, the only interaction forces between two particles are friction forces and repulsion on contact. In DEM with soft particles, the particles may overlap which is exploited in the force calculation. The repulsive normal force is computed using the geometry of the overlap: it is taken proportional to the overlap area of the two colliding particles. A distance mimicking elastic displacement is calculated from this area and a characteristic length, which is essentially proportional to the distance of the centres of mass of the particles. Its precise definition is given in [17]. In order to have a realistic simulation, the overlap distance must be small enough in comparison with the particle diameter. To make collisions inelastic, a dissipative normal force is included in our algorithm. The calculation of the tangential force F_t is more complicated than that of the normal force F_n, because it is to be determined from the Coulomb condition $|F_t| \le \mu F_n$, where μ is the static friction coefficient; this is just an inequality, not an evaluable formula. A detailed analytic description of both normal and tangential forces may be found in [17].

Particle centres of mass are put on a regular triangular or rectangular grid (with basis, i.e., centres of rectangles are grid points, too) so that initially particles are close to each other but do not touch. For monodisperse particles, very small distances can be chosen, since the particle shapes fit in circumscribed circles of the same size. Then the particles are released under gravity to drop either on a rough horizontal bottom (made up of particles itself) or a smooth straight one. The system is delimited by vertical walls on both sides. The simulated rectangular system then consists of several thousand roundish but polygonal particles.

A picture of a system containing monodisperse particles is shown in Fig. 1, where the particles have been deposited on a smooth bottom; the starting configuration was triangular, so each particle has six nearest neighbours in this very regular arrangement. (In the rectangular arrangement discussed below, each particle has four contacts only.)

The aspect ratio of the aggregate is approximately 1:4 for the monodisperse packing, and we also consider the same aspect ratio for the other packings. This size of the aspect ratio is usually needed in order to study the response function with negligible effects from the lateral system boundaries.

Once the assembly is ready, we apply a load (using a piston) to a single grain at the top surface of the system. This force is vertically downward. The arrangement is displayed in Fig. 1. The external load (point force) is small enough so as to not cause any rearrangement of the layer structure, which means that neighbourhood relationships between particles defined by their contacts do not change during loading. In fact, the first interesting question is that of the *linear response* of the system, which would impose even more stringent bounds on the smallness of the external force. From this linear response, inferences on the linear part of the underlying continuum equations can be made. Nonlinear effects are of course of interest, too, but response functions are insufficient for their characterization.

We have checked that the loading does not lead to any rearrangements of the packing. We take a piston velocity of $0.01\, m/s$, which turns out to be small enough. As shall become transparent later, linearity of the response was not always satisfied, with some interesting consequences. We choose a static friction coeffient $\mu = 0.5$ for the packings of monodisperse, bidisperse and pentagonal particles. The height of the rectangular packing is 15 cm.

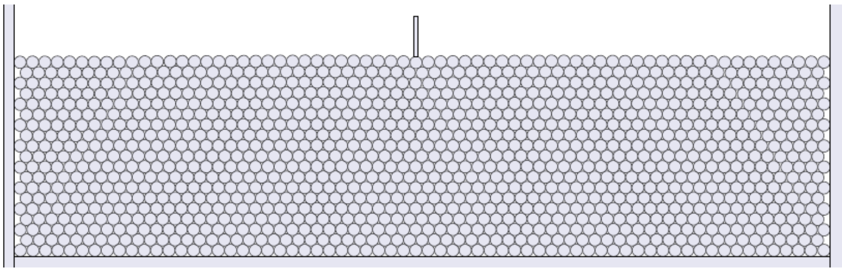

Figure 1: Rectangular packing of monodisperse particles on a smooth bottom with a load applied to a single grain at the top surface of the system.

3 Contact force angle distribution

A first microscopic characterization of the rectangular assemblies is carried out computing the contact force angle distribution. The polar diagrams of the angular distribution of all contact forces for different simulated packings are displayed in Fig. 2. All angles are measured with respect to the horizontal axis. Ostensibly, the orientation distributions of the contact forces are different for the different packings. The monodisperse packing is highly ordered, because the forces are mostly oriented along a few fixed directions given by multiples of 60 degrees.

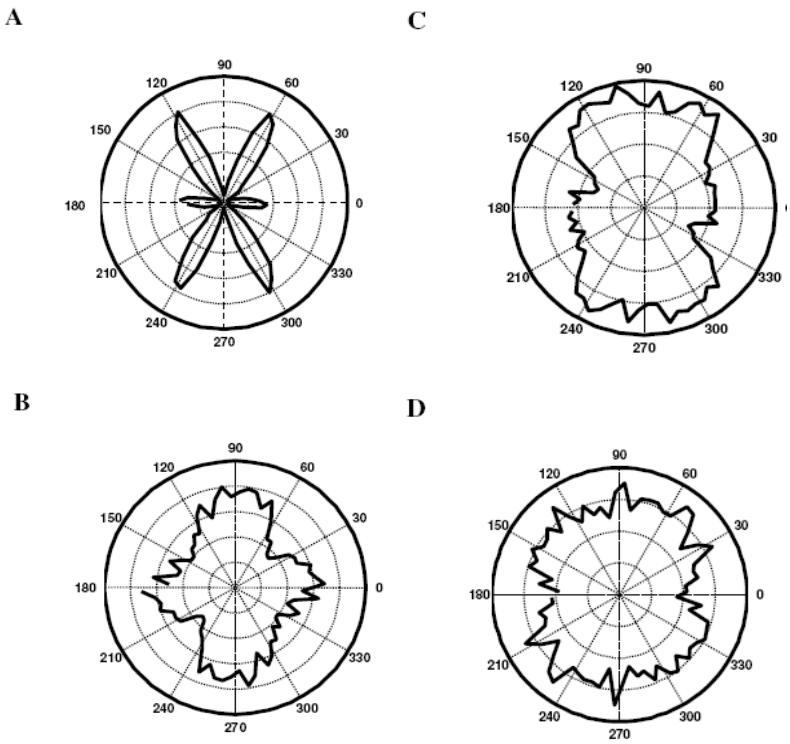

Figure: 2 Contact force angle distribution of the particles for the different packings. **A,** for monodisperse packing**. B,** for bidisperse packing. **C,** for pentagonal packing. **D,** for polydisperse packing.

The contact disorder of the remaining packings is increasing from the bidisperse packing via the pentagonal to the polydisperse packing. For polydisperse packing, the sample is highly disordered, since the angles of the contact forces almost form an isotropic orientation distribution. Note that in all the distributions the direction of gravity has a definite influence on the shape of the distribution; more forces have large components parallel to the vector of gravitational acceleration than perpendicular to it. This influence is weakest for the polydisperse system.

4 Simulation results on stress responses for different packings

Once we have the forces and their points of contact, we can determine a formal stress tensor of a single particle, and then it is easy to determine the average of the stress tensor over many particles in a representative volume element (REV). An REV is defined by the requirement that the averaging procedure gives an unchanged result on increase of the volume element. We have determined the necessary size of a few hundred particles in preceding numerical experiments [19]. The stress response function is obtained by taking the difference of the stress distribution measured on the system with additional external load and the stress distribution of the same system without load.

$$\sigma_{ij}(x,y) = \sigma'_{ij}(x,y) - \sigma''_{ij}(x,y). \tag{2}$$

Here, $\sigma'_{ij}(x,y)$ and $\sigma''_{ij}(x,y)$ are the averaged stress tensors at position (x,y) with and without external load, respectively. Averages were taken over nine realizations of the rectangular aggregate in order to suppress fluctuations present within a single realization.

We present our simulation results for the average vertical normal stress response at different heights of the sample consisting of monodisperse particles in Fig. 3 (*h* is a height measured from the bottom of the rectangle up, depth would be measured from its top down). It can be seen that at a large depth below the perturbing force, the vertical normal stress response has a double peak, a behaviour which is predicted by hyperbolic continuum equations such as those from the model of oriented stress linearity (OSL) [1]. For small depths, these two peaks merge into a single peak. In the next figure, Fig. 4, we plot the vertical normal stress response for the bidisperse packing at the same heights of the sample as for the monodisperse packing. The double peaks are also present in this case, but much less pronounced than for the monodisperse packing. Fig. 5 shows the simulation results of vertical normal stress response for the pentagonal packing at different heights of the sample. The vertical normal stress response shows no evidence for wave-like stress propagation. It has the form of a bell-shaped curve with a single peak within each layer of the sample and the width of the response increases slowly with the distance from the perturbation point.

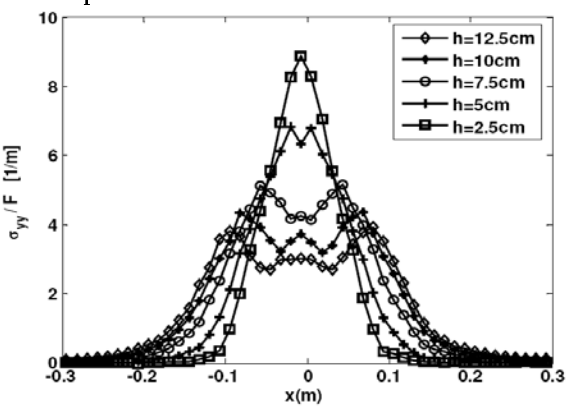

Figure 3: Vertical normal stress response at different depths of a system consisting of a monodisperse assembly of particles.

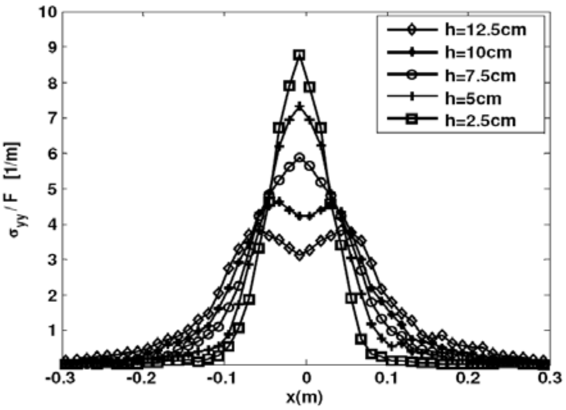

Figure 4: Vertical normal stress response at different depths of a system consisting of a bidisperse mixture of particles.

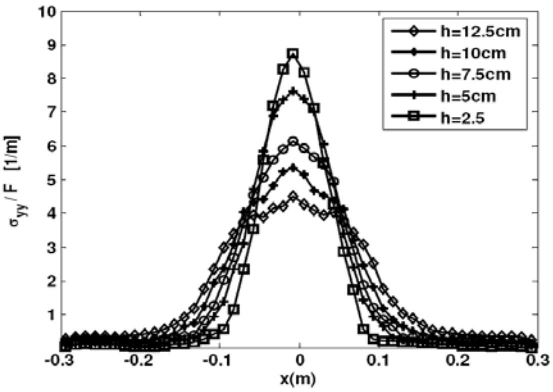

Figure 5: Vertical normal stress response at different depths of the system consisting of an assembly of pentagonal particles.

Comparison with the available experimental results by Junfei Geng et al. [8] shows that the stress responses reproduce the experimentally observed behaviour for the packing of monodisperse, bidisperse and pentagonal systems.

5 Stress responses for different values of the friction coefficient

In this section, we are interested in determining the averaged vertical normal stress response of a rectangular system made from monodisperse particles for different values of the coefficient of static friction. We prepare the sample by arranging roundish particles on a rectangular lattice, so that neighbouring particles in the same layer have a larger distance than to their nearest neighbours in the two layers below and above (the distance to which was chosen extremely small – about 0.01 percent of the particle diameter). After dropping and relaxation, each particle in the bulk has a fixed number of four contacts in this rectangular packing. Three different series of rectangular samples are created using the same material and the same simulation parameters except for the friction coefficients μ, for which the values $\mu = 0.3$, $\mu = 0.6$, and $\mu = 0.9$ were taken, respectively.

The simulation results for the stress response of a rectangular packing consisting of a monodisperse arrangement of particles for different friction coefficients μ are displayed in Fig. 6. For small values, such as $\mu = 0.3$, the response consists of two peaks (Fig. 6 A). This doubly peaked shape appears not only at the greatest depth, but also exists up to a rather higher level of the sample. Another interesting aspect of this graph is that the stress response dip is very large in this case.

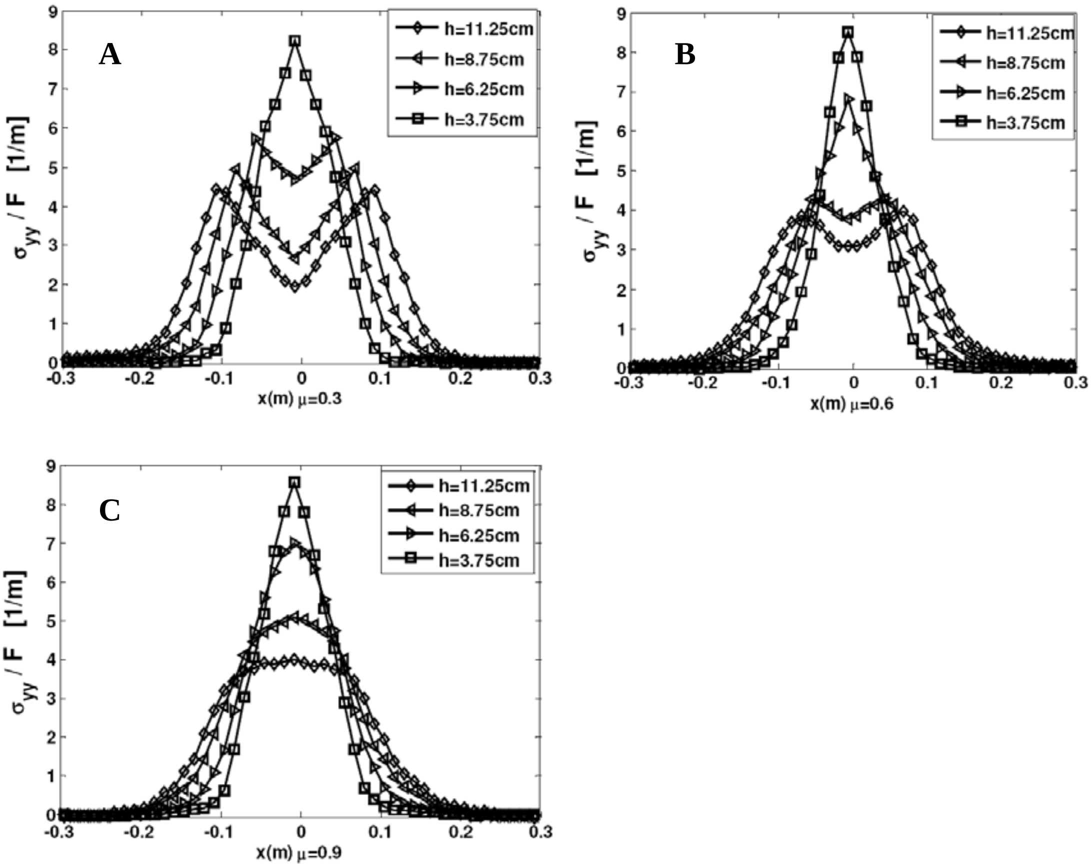

Figure 6: Simulation results of vertical normal stress response for different values of the friction coefficient. **A**: $\mu = 0.3$, **B**:, **C**: $\mu = 0.9$.

For $\mu = 0.6$, the response also consists of a doubly peaked shape up to a certain height of the sample, as shown in Fig. 6. B, but the response dip is much less pronounced and the two peaks merge into a single one when the distance from the point of the application of the external load becomes small. For $\mu = 0.9$, the response consists exclusively of a single central peak at each height level of the sample as shown in Fig. 6 C, with a width that broadens with increasing depth. In conclusion, the depth of the crossover between double- and single-peak response decreases with increasing μ. That means when μ is very large, the response consists of just a single peak, but when μ is very small, the response has two peaks.

To explain the transition from two peaks to one, we recall that for the case of large values of friction coefficient, the frictional contacts will be only partially and randomly mobilized which increases the *force* disorder in the system even if it is geometrically rather ordered. In the case of small values of the friction coefficient, most of the frictional contacts are fully mobilized which means that there is little static indeterminacy (much of the system is at the threshold of sliding) and forces are entirely determined by the geometrical configuration. If the configuration is ordered, so are the forces. We will elaborate a bit more on this point below.

First let us discuss another effect of friction. We determine vertical normal stress responses of monodisperse systems with different coefficients of static friction for different values of a vertically applied force, with the particles initialized on a triangular lattice. The simulation results for just the bottom layer of the system are displayed on Fig. 7 A for the case of a static friction coefficient $\mu = 0.3$

and different external loads. There is a transition from a single peak to a doubly peaked structure as the applied load is increased. On the other hand, when μ is sufficiently large ($\mu = 0.8$), the stress response consists of a single peak for the same amount of external loading, as is shown in Fig. 7 B.

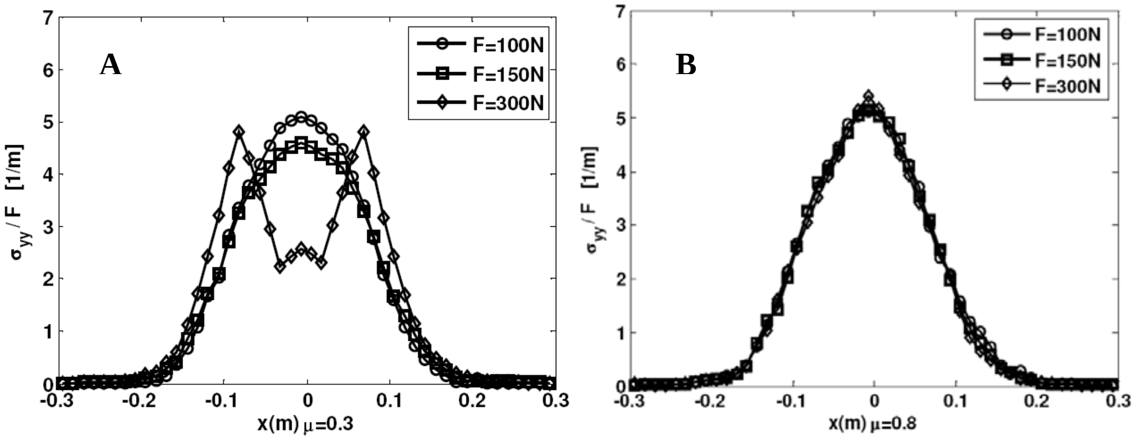

Figure 7: Vertical normal stress response for different values of applied loads at the bottom layer of the rectangular assembly. A: static friction coefficient $\mu = 0.3$, B: static friction coefficient $\mu = 0.8$.

Note that on the right-hand side, the shape of the stress response function does not change as the external force is increased. Since the stress component is normalized with respect to the external force, this means that the response is *linear*. On the other hand, the left-hand panel then obviously corresponds to a nonlinear response. How can this be interpreted?

In the system with large μ, most contacts are not mobilized, i.e., the friction force is smaller than μN, where *N* is the normal force at the contact. When the external force is increased, the force at the contact can rise proportionally to it for some time. For small μ, many contacts will be almost mobilized, hence on increase of the external force they will become mobilized and sliding will set in. Since there are no substantial particle rearrangements yet in the force range considered, this must stop again due to geometric constraints, and there will be *force rearrangements* instead, destroying the proportionality of the force network to the external force. (It should be mentioned that in our simulations the static and dynamic friction coefficients are taken equal, so there is no hysteresis between sliding and sticking.) In the force network at the increased external force, more contacts will be mobilized meaning that the corresponding forces are determined by the local contact geometry alone. Since the geometry is pretty ordered in the monodisperse system, this should correspond to a *force ordering* as well. Reduced disorder in the force distribution will then lead to wave-like stress propagation and a double peak.

To see wether these ideas are right, we compute a histogram of the force distribution, more precisely of the distribution of force magnitudes. This is given in Fig. 8. We bin the forces with norm values inside intervals of a fixed size and plot the numbers so obtained as boxes labeled by the norm itself. On the left hand side of the figure, this is done for *F*=100 N, on the right hand side for *F*=300 N, at $\mu = 0.3$, i.e. for the case given by Fig. 7 A. To make things comparable, the bin size has been taken three times larger for *F*=300 N than for *F*=100 N. It is obvious from the graph that the relative width of the force distribution is much smaller for the larger external force, meaning indeed increased order in the force network.

What might be considered surprising is that in spite of the still appreciable level of disorder in the large force case – after all the distribution of forces is not extremely sharp – the transition from elliptic to hyperbolic redistribution of stress already takes place. But clearly one has to consider not only the sizes of the forces but also their directions, and the orientation ordering discussed in Sec. 3, reinforcing stress propagation along preferred paths (which is typically describable by hyperbolic equations [1,3]) is also increased when friction contacts become fully mobilized, because the resultant of the friction force and the normal force then makes a fixed angle with the latter in one of two preferred directions.

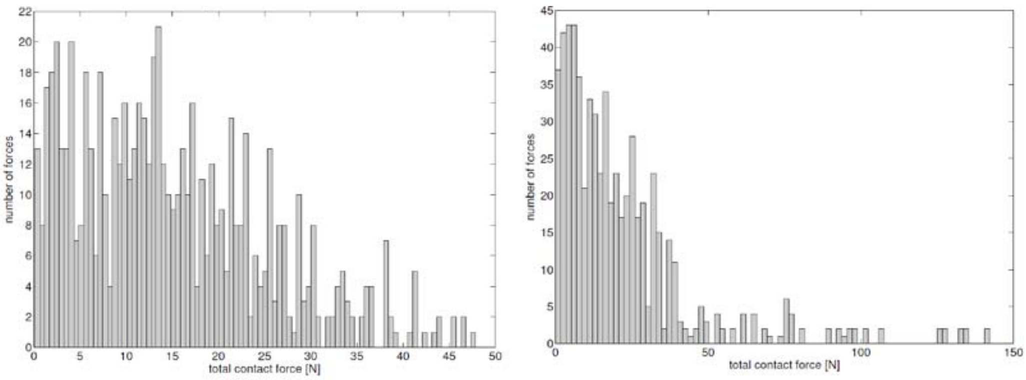

Figure 8: Left: force distribution in the monodisperse assembly with static friction coefficient $\mu = 0.3$ for an external force *F*=100 N. Right: the same for *F*=300 N.

An interesting corollary of these considerations is that increasing static friction increases the range of elastic behaviour of the granular aggregate – the range of linear response is bigger for larger μ and the response is non-propagative, suggesting equations of motion describing essentially linear elasticity.

6 Stress responses for polydisperse packing

In this section, we focus on determining the stress response for a system containing a polydisperse mixture of particles, then on comparison of simulation results for the stress response with existing experimental results [11]. For polydisperse packings, we prepare samples with rough as well as with smooth bottoms. In order to prepare a sample with a smooth bottom, we use a flat bottom plate with friction on which the particles are dropped, whereas for a rough bottom the "plate" consists of a set of spatially fixed particles of the same size as the filled-in particles, in order to simulate surface asperities.

Simulation results on the vertical normal stress response at different heights of the system containing a polydisperse mixture of particles with smooth bottom are shown in Fig. 9. As can be seen in the figure, the stress response consists of a single peak at all height levels investigated. We did not observe two separate bumps as predicted by the hyperbolic models of refs. [12,20]. Hence, it is to be recognised that generally for large contact disorder the vertical normal stress response consists of a single peak only, which means that it is an "elastic-like" response, a behaviour describable by elliptic equations rather than wave equations. Moreover, the width of the response increases linearly with the distance from the perturbing force. We compare our simulation results qualitatively with existing experimental results shown on the right hand side of Fig. 9, obtained by G. Reydellet et al. [11] using "aquarium sand" and "Fontainebleau sand" and a poly-dispersity of about 50%. Comparison shows that our simulation results are in very good agreement with the experimental results.

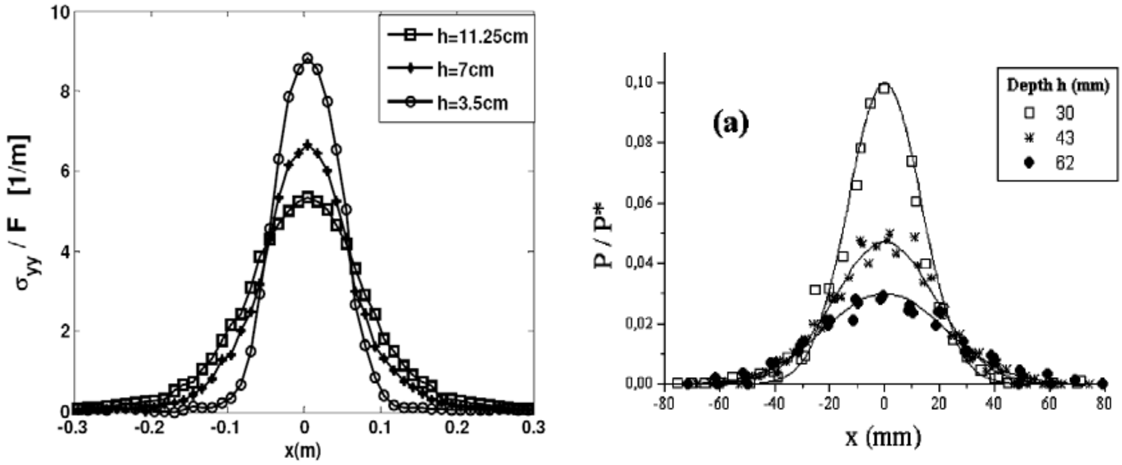

Figure 9: Vertical normal stress responses to a point source for a rectangular assembly of granular material with a smooth bottom, consisting of a polydisperse mixture of particles. The left-hand side of the figure shows simulation results, whereas on the right hand-side experimental results are displayed [11].

In the next step**,** we compare the vertical normal stress between the rough and the smooth bottom cases with the same materials. The stress response for the system containing a polydisperse mixture of particles with a rough bottom obtained from the simulation is represented in Fig. 10. In this case, the stress response consists of a single peak, too. We notice that the behaviour of vertical normal stress response does not show much difference between the two systems (Fig. 9, left, and Fig. 10). The stress response for a rough bottom is about 7% smaller than that for a smooth bottom.

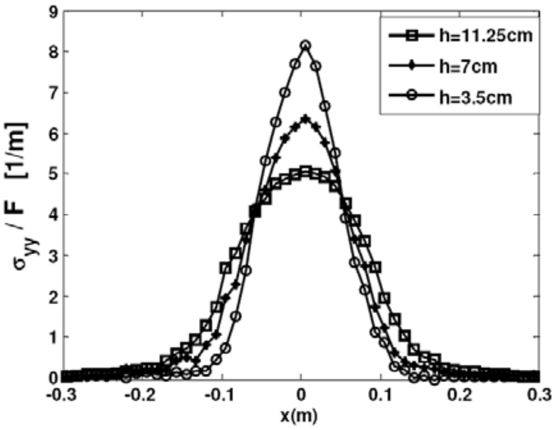

Figure 10: Vertical normal stress responses to a point source for a rectangular assembly of polydisperse granular material with a rough bottom.

7 Comparison between numerical stress responses and an analytic solution

While a comparison with experiments, for which we have no theory, only shows the viability of our code, a comparison of stress responses from the numerics with an analytic solution in the framework of isotropic linear elasticity for a semi-infinite 2D medium (a half plane, $y > 0$) will give an idea about the validity range of these concepts in granular media. According to Boussinesq and Cerruti [21], an analytic expression for the stress tensor components with a force F applied at a point (x=0) on the edge (y=0) of the half plane reads:

$$\sigma_{yy} = \frac{2Fy^3}{\pi(x^2+y^2)^2}. \tag{3}$$

In Fig. 11, we compare the results of the vertical normal stress response of a semi-infinite system at a distance corresponding to that of the bottom layer of our finite system with the responses from the numerical simulation for the rectangular system of a polydisperse mixture of particles with either a rough or a smooth bottom. The height of the rectangular system is 15 cm. We have taken averages over twelve realizations.

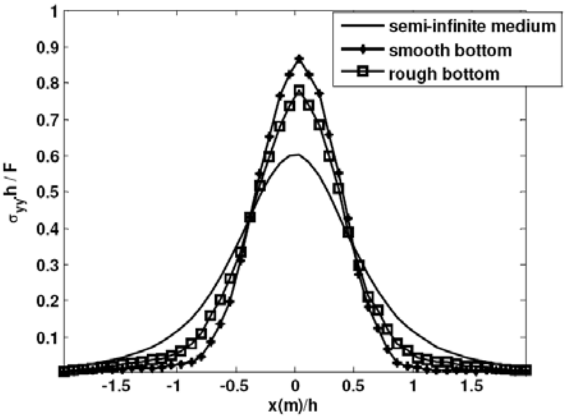

Figure 11: Simulation results for the vertical normal stress response of a rectangular assembly, at the bottom layer for two different systems, compared with an analytic solution.

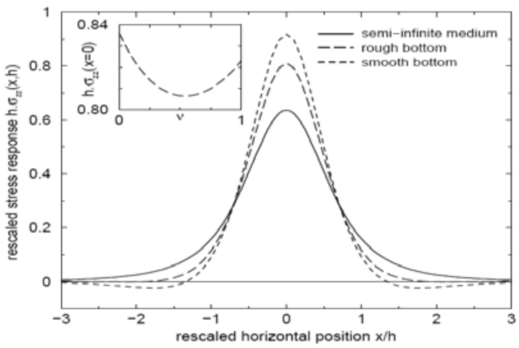

Figure 12: Experimental results of vertical normal stress response of rectangular assembly at the bottom layer for three different systems [9].

In Fig. 11, the solid black curve shows the analytic result, whereas the line connecting squares and the one connecting crosses represent the simulation results for a rough bottom and a smooth bottom, respectively. The figure demonstrates that the vertical normal stress responses for the three cases are qualitatively similar. In addition, there is no double peak below the point where the external force is applied. The experimental results of ref. [9] obtained using a polydisperse mixture of "Fontainebleau" sand grains are represented in Fig. 12. It can be seen that our numerical results agree well with the experiments.

8 Strain response for monodisperse packing

We use a general differentiation method applied to particle displacements to determine the vertical incremental normal strain tensor for the rectangular aggregate consisting of a monodisperse arrangement of particles, where the particles have been placed on a triangular lattice above a rough bottom. Once we have the individual particle displacements we can calculate the strain tensor. The displacement vector of a single particle reads

$$u_i = x_i' - x_i, \tag{4}$$

where x_i is the initial position of the particle i and x_i' is the final position of the particle after applying the load to the sample. Then, we take an average over individual displacements of the particles inside the REV to determine a continuous displacement field as a function of the point position (x, y)

$$u_x(x, y) = \frac{1}{n}\sum_{i=1}^{n} u_i. \tag{5}$$

The sum is over the particles in a box centered at (x, y). This procedure allows us to determine the components of the strain tensor at the point considered. The simplest approximation for the vertical component of the strain tensor (which generally is a linear combination of derivatives of the displacements) is as follows

$$u_{yy}(x, y) = \frac{u_y(x, y + h_y) - u_y(x, y)}{h_y} \tag{6}$$

The vertical strain tensor is averaged over many systems in order to eliminate the fluctuations of the single system. In Fig. 13, we plot the vertical normal strain responses at different depths of the packing that contains monodisperse particles. The strain response has the form of a bell-shaped curve with a single peak and the width of the response increases with the distance from the perturbation point. It is noted that for monodisperse packing the strain response displays a peak structure different from that of the stress response. This result demonstrates the failure of isotropic elasticity in this rather singular, because ordered system.

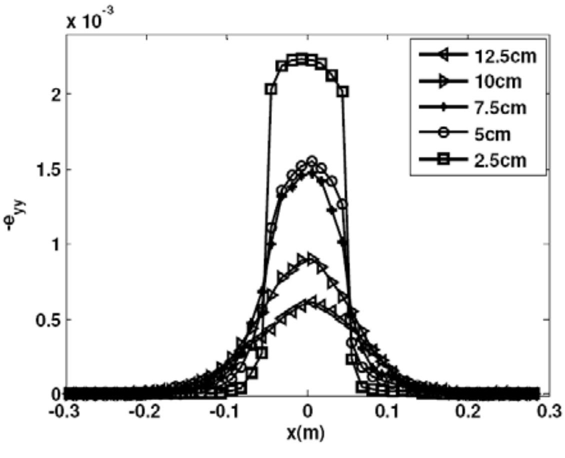

Figure 13: Vertical normal strain response for the packing that contains monodisperse mixture of particles.

9 Conclusions

To conclude, we have determined the stress response of 2D granular packings to a local force perturbation. We observe that for packings with strong spatial order, the average stress response shows a behaviour corresponding to that of hyperbolic continuum equations like those of the OSL model. As the amount of contact disorder increases, there is no wave-like stress propagation anymore, and behaviour emerges that one would expect to arise from a description in terms of elliptic equations. Comparison with the experiments performed in [8] shows that the vertical normal stress responses for different packings are qualitatively similar to experimental ones.

We observe that both the static friction coefficient and the external load affect the stress response of rectangular assemblies of granular materials. A quite spectacular effect is the force ordering on decrease of the static friction coefficient, leading to a switch-over from an elliptic response to a hyperbolic one and to nonlinear response well below the level of plastic rearrangement. It should also be noted that except in the case of a completely ordered system (which in practice is never realized) the response to *infinitesimal* perturbations should always be describable by elliptic equations. Hence, any set of constitutive equations for a granular material trying to capture macroscopic elasto-plastic behaviour must have the property that the small-load and small-displacement limits are elliptic. On the other hand, the equations may (and in some cases should) turn hyperbolic for large load amplitudes. Writing such a set of equations is a challenge well beyond the scope of this article.

In addition, the vertical normal stress response definitively reveals elastic-like stress responses for system constructed from a polydisperse mixture of particles with a rough and a smooth bottom, and the behaviour of the two systems is qualitatively similar.

We compare the analytic vertical normal stress solution of an isotropic linearly elastic semi-infinite medium in 2D with our numerical stress responses for both smooth and rough bottoms. We find that the vertical stress response is qualitatively similar for the three different systems. Comparison with the available experimental results from [9] shows good agreement. We have determined the vertical normal strain response for the monodisperse packing and find it to display a peak structure different from that of the stress response, suggesting that if linear elasticity is applicable – at small loads –anisotropy effects have to be taken into account. Isotropic linear elasticity is not good enough.

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