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Influence of the geometry on the pressure distribution of granular heaps

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Abstract We investigate the effect of the geometry of granular heaps on the pressure distribution. For given pressure distributions under cones we compute the pressure distribution under wedges using linear superposition. For cones with a pressure minimum, the pressure minimum for the corresponding wedge vanishes. Comparisons with experimental data gives good qualitative aggreement, but the total pressure is overestimated.

1 Introduction

In recent years, pressure distributions under heaps of dry grains have received much attention [1–8] due to the pressure minimum which was observed experimentally in granular cones without bed displacements [9–13]. In granular wedges, no [14, 15] or only small [9] pressure minima could be observed. Brockbank and Huntley [13] found experimentally a pressure dip in cones, except for large (non-cohesive) polydisperse particles. In granular matter, many effects such as the grain's material, Young's modulus, surface roughness, coefficient of friction and history of the construction influence the properties of heaps [10, 16]. Due to the rich phenomenology, the pressure distribution under a heap has become a kind of paradigm in granular media research. In this paper, we focus on the effect of the shape of sand-heaps on the pressure distribution by taking into account only the geometry of the heaps.

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2 Overview

To simplify matters, we will assume a homogeneous density of the heaps, and we also assume that the mass density in the cone is the same as the mass density in the wedge. Later on, we will question this assumption in the comparison with experimental data, but for the simplicity of the derivation the densities will be set to unity.

Because experimentally the angle of repose is practically the same for wedges as for cones, we do not use the angle of repose α,

$$\frac{h}{r} = \tan\alpha \tag{1}$$

of the heap of hight h and base length $2r$ explicitly.

We assume that when we go from a cone to a wedge, there is neither a change of the constitutive equations, nor in the internal structure and only geometric effects come into play. We use a phenomenological approach to compute the contours and pressures via linear superposition under the assumption that "nothing else" changes, see Fig. 3. This approach of "all things being equal" eliminates effects from e.g. the constitutive equations, the material parameters or the micro-structure and we only observe effects based on the geometry. We do not propose that the situation is really as simple as that. The superposition principle for pressures itself is valid also for linear homogeneous partial differential equations (PDE's). PDE's have been widely used in theories on the stresses in static granular materials [17–20]. Using the superposition principle, we derive why no significant pressure dip7 can be found under granular wedges with non-deformed bottoms [16], though there are several experiments with pressure dips under granular cones [9–13].

In the next step, we apply this method to experimental data. We compare the pressure measurement under a wedge with the theoretical prediction from our theory which uses the cone pressure for the same material as starting point. We obtain the correct qualitative pressure distribution, but the theory overestimates the total weight.

3 Superposition of cones

In this section, we show how to integrate over the contour of a cone in order to obtain the contour of the wedge, and

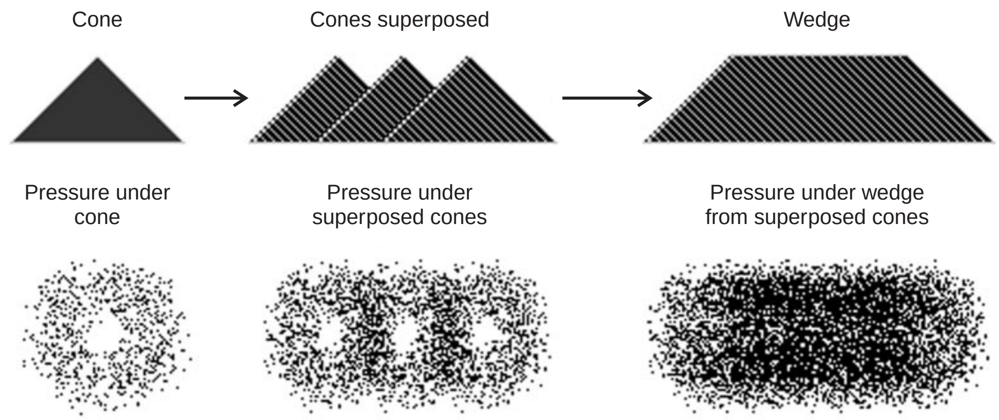

Fig. 1. Building a wedge out of cones for the heap shape (above) and the corresponding pressure distribution (below). The resulting pressure is given as the intensity of the point clouds

which normalizations are possible to conserve either the height or the volume.

3.1 Integration of cones

The contour of the heap is position-dependent and (see Fig. 3) has the dimension of a length. We will denote the maximum height of the heap as h. If the contour C^{cone} of a cone with diameter $2r$ and maximum height h is given as $C^{\text{cone}} = C^{\text{cone}}(x, y)$, one can obtain the contour C^{wedge} of a wedge (see Fig. 2) with base width $2r$ by superposition as

$$C^{\text{wedge}}(y) = \frac{1}{2r} \int_{-\sqrt{1-(y/r)^2}}^{\sqrt{1-(y/r)^2}} C^{\text{cone}}(x, y) dx. \tag{2}$$

$C^{\text{wedge}}(y)$ depends only on y and not on x due to the translation invariance of the wedge along the x-axis. Eq. (2) is an average over the contour along the x-direction.
The prefactor in Eq. (2) $\frac{1}{2r}$ enters in front of the integral, because the units of the integration increment dx must be compensated. In terms of units, a prefactor $\propto \frac{1}{r}$ is the only possibility to fulfill the condition that the contour of

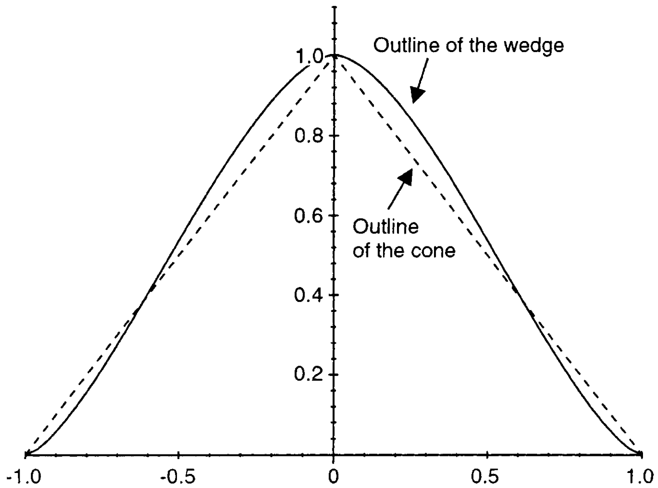

Fig. 3. Outline of a cone with straight slopes and a wedge constructed by infinitesimally adding cones with straight slopes in a gedanken experiment

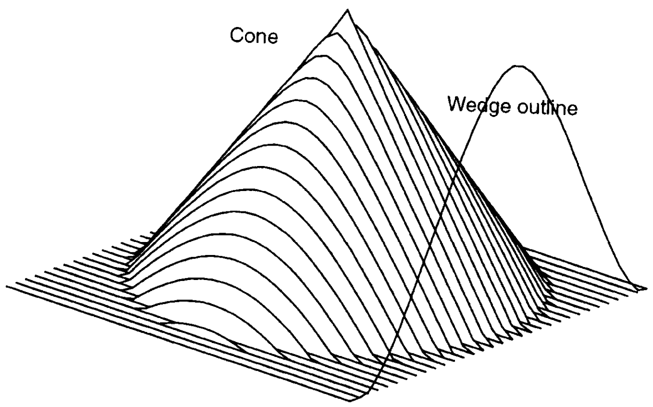

Fig. 2. Sketch for the integration process for a cone: Averaging over the hight distorts the straigt slopes of the cone

the wedge has the same units as the contour as the cone, which, as has been mentioned above, has the dimension of a length. The dimensionless prefactors to preserve the mass or the height of the heap will be discussed in section 3.2.

The integration bounds $\pm\sqrt{1-(y/r)^2}$ in Eq. (2) result from the circular base of the cone and depend on the geometry at the base of the heap. For the integration of a pyramid with a square base of $2r \times 2r$, the integration would range from $-r$ to r. (see subsection 3.3 and Fig. 6). In the following text, we take the radius of the cone (and therefore the width of the wedge) to be $r = 1$ for simplicity, so

$$x/r \to x \tag{3}$$
$$y/r \to y \tag{4}$$
$$h \to r \tag{5}$$
$$C^{\cdots}(x, y)/h \to C^{\cdots}(x, y). \tag{6}$$

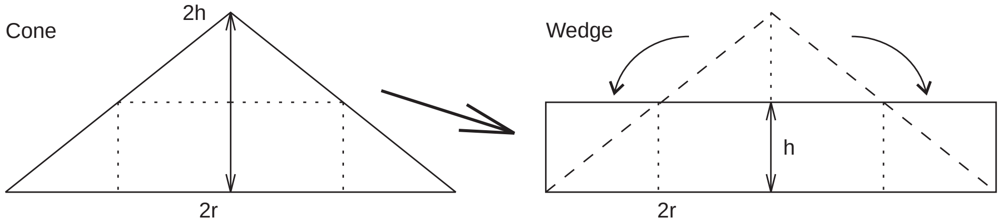

Fig. 4. Averaging the cone to a wedge with normalization factor $1/2r$ reduces the height of the heap from 2h to h in the middle of the wedge

The old names will be retained for the new quantities. Any prefactors for the height of the heap or the angle of repose will be omitted, because due to the normalizations in section 3.2, the height of the wedge is given automatically by the height of the corresponding cone. All functions for the contours and pressure will be 0 outside the heap.

If a wedge is built by infinitesimally integrating cones with straight slopes,

$$C^{\text{cone}}(x, y) = 1 - \sqrt{(x^2 + y^2)} \tag{7}$$

the profile of the wedge $C^{\text{wedge}}(y)$ can be described as

$$\begin{aligned} C^{\text{wedge}}(y) &= \frac{r}{h}\sqrt{1-y^2} - \frac{y^2 \ln(\sqrt{1-y^2}+1)}{2} \\ &\quad + \frac{y^2 \ln(-\sqrt{1-y^2}+1)}{2}, \end{aligned} \tag{8}$$

which is plotted in Fig. 3. This seems to be still a reasonable approximation for the slopes of a "wedge". This wedge constructed from a cone with straight slopes has no straight slopes, because one has to integrate different conical sections at each y. "On average", the critical angle is nevertheless preserved.

3.2 Possible normalizations

Eq. (2) corresponds to an average along the x-axis and the normalization of $1/(2r)$ conserves the cross-section area along the x-axis of the cone. As this cross-section is triangular, but the section along the x-axis of a wedge is rectangular, the height of the wedge will be only half of the height of the cone, see Fig. 4. One has to choose the prefactors depending on whether one wants to preserve the height of the cone or the mass per volume. Both are different because the cross section of the cone does not coincide with the cross section of the wedge, see Fig. 3.

In the following sections, we will use the prefactor $1/r$ for wedges instead of $1/(2r)$ for Eq. (2) so that the wedge has the same height as the cone. This leads to a normalization factor of 2 in from of the integral of Eq. (2). As can be seen by Fig. 3, the geometry is not preserved, therefore the mass is not conserved properly. To guarantee, that

$$\frac{M^{\text{wedge}}}{M^{\text{cone}}} = \frac{V^{\text{wedge}}}{V^{\text{cone}}} \tag{9}$$

for the pressures, one has to choose the normalization factor not as 2 but as

$$\frac{V^{\text{wedge}}}{V^{\text{cone}}} = \frac{2r^2 h}{\pi/3 r^2 h} = \frac{6}{\pi}. \tag{10}$$

This leads to a normalization factor of $\frac{6}{\pi}$ instead of 2 in comparison to the normalization with the height of the heap.

3.3 Superposition of non-conical heaps

The only geometric figure which in an averaging procedure such as Eq. (2) preserves the triangular cross section is a wedge with straight slopes and square bottom (Fig. 5). The integration of pyramids (Fig. 6) with a square basis leads to a "wedge" with parabolic slopes, as can be seen in Fig. 7. The integration limits in Eq. (8) must be changed, so that

$$C^{\text{pyr.wedge}}(y) = \int_{-1}^{1} C^{\text{pyramid}}\, dx \tag{11}$$

$$= \int_{-1}^{1} 1 - \max(|x|, |y|)\, dx \tag{12}$$

4 Superposition of pressures

The pressure distribution can be obtained by superposition in the same way as the contours. We investigate the superposition of different pressure curves to see how the change from the cone to a heap affects the pressure distribution.

The x-dependence of the pressure in the wedge can be dropped due to the translation invariance like for the contours. For simplicity, we take the radius of the cone (and

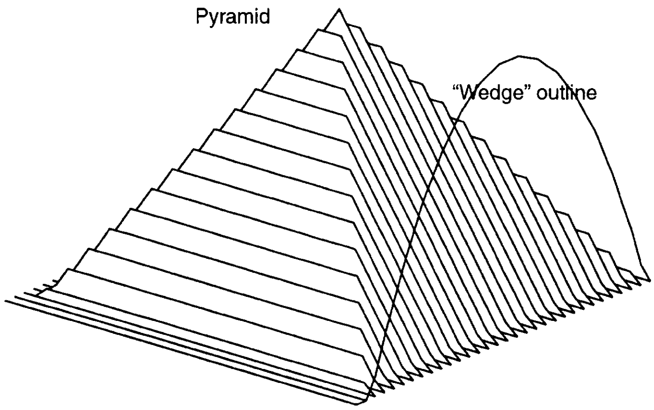

Fig. 5. Only the integration of a wedge with straight slopes again yields a wedge with straight slopes

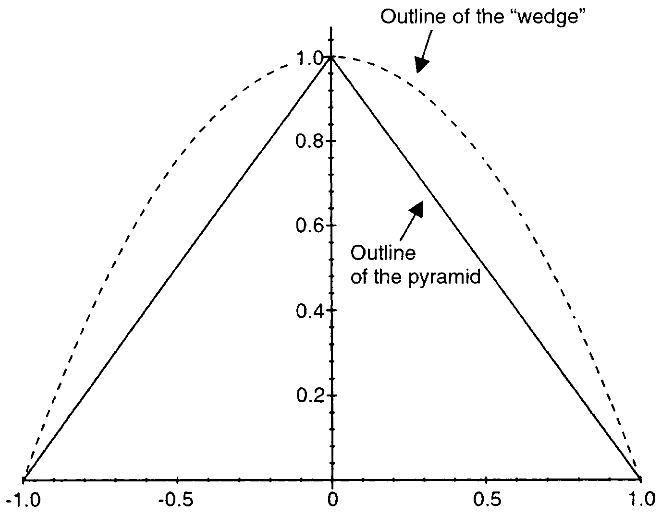

Fig. 6. Sketch for the integration process for a pyramid

therefore the width of the wedge) to be $2r = 1$. Any prefactors for the normalization of the heap will be omitted for the qualitative discussion in this section.

If the pressure under a cone with radius r is given as $P^{\text{cone}} = P^{\text{cone}}(x, y)$, one obtains the pressure under the corresponding wedge by linear superposition as

$$P^{\text{wedge}}(y) = 2 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} P^{\text{cone}}(x, y)\, dx. \tag{13}$$

This relates the total pressure under the cone to the total pressure under the wedge, so that the masses computed via the pressure distribution are also directly related.

Different pressure distributions represent different materials. At least in two dimensions, different "building histories" of the heap also crucially influence the pressure distribution [8]. The integration of the heap contour itself is a purely geometric concept which results from the superposition principle for PDE's and is in no way related to a "physical" building up of the heap.

4.1 Masses of the heap

The masses of the heap can be derived via the pressure on the ground. The ratio for the volume of a cone to a wedge is 1 : 2 for the height normalization in Eq. (13). Because we assume that the density in the wedge is the same as the density in the cone, 1 : 2 is also the ratio between the masses of the cone to the wedge. This must also be true for the masses as computed via the bottom pressures. The mass of the cone is

$$M^{\text{cone}} = \int_{y=-1}^{y=+1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} P^{\text{cone}}\; dx\; dy \tag{14}$$

The mass of the wedge is

$$M^{\text{wedge}} = \int_{y=-1}^{y=+1} \int_{x=-1}^{x=+1} P^{\text{wedge}}(y)\, dx\; dy \tag{15}$$

$$= 2 \int_{y=-1}^{y=+1} P^{\text{wedge}}(y)\mathrm{d}y \tag{16}$$

$$= 2 \int_{y=-1}^{y=+1} \int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} P^{\text{cone}}(x, y)\, dx\; dy \tag{17}$$

$$= 2M^{\text{cone}}. \tag{18}$$

Therefore, for linear superposition the proportion between the total weight of the wedge and the cone $M^{\text{wedge}}/M^{\text{cone}}$ is independent of the detailed pressure distribution. The derivation from the masses from the volumes is analogous, only the pressures have to be replaced by the product of the volume and the density.

4.2 Zero pressure under the apex of the cone

In the following, we will use the Heaviside function

$$\Theta(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0, \end{cases} \tag{19}$$

and the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\mathrm{d}t. \tag{20}$$

If one chooses a pressure distribution for the cones constructed from triangles (similar to the one proposed in [20] for two-dimensional heaps, see Fig. 7) according to

$$\begin{aligned} P_1^{\text{cone}}(x, y) = {} & |y|\, \Theta(-y + 0.5)\Theta(y + 0.5) \\ & + (1 - |y|)\,(1 - \Theta(-y + 0.5)\Theta(y + 0.5))\,, \end{aligned} \tag{21}$$

the pressure distribution of the wedge will be

$$\begin{aligned} P_1^{\text{wedge}}(y) = {} & \sqrt{1 - y^2} - 0.5y^2 \ln(\sqrt{-(y-1)(y+1)} + 1) \\ & + 0.5y^2 \ln(-\sqrt{-(y-1)(y+1)} + 1) \\ & - 0.5\Theta(-y + 0.5)\Theta(y + 0.5)\sqrt{1 - 4y^2} \\ & + y^2 \ln(0.5\sqrt{-(2y-1)(2y+1)} + 0.5) \\ & \times \Theta(-y + 0.5)\Theta(y + 0.5) \\ & - y^2 \ln(-0.5\sqrt{-(2y-1)(2y+1)} + 0.5) \\ & \times \Theta(-y + 0.5)\Theta(y + 0.5). \end{aligned} \tag{22}$$

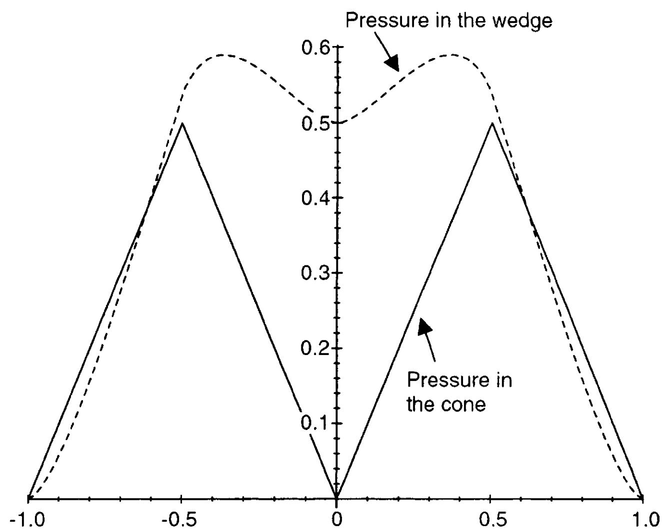

Fig. 7. Outline of the pyramid with square base and the "wedge" constructed from that pyramid via integration

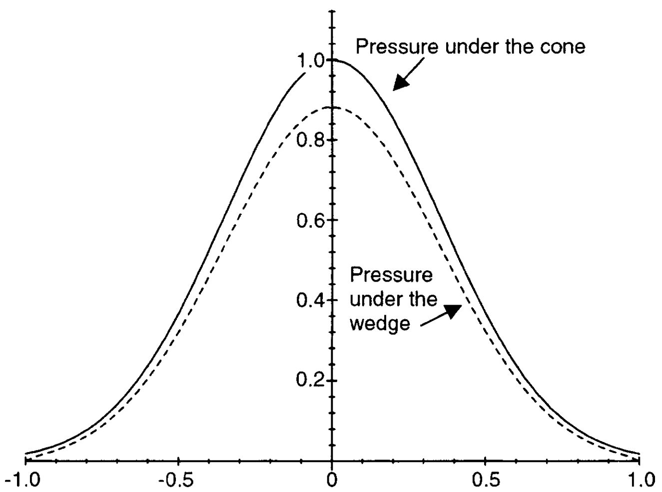

Fig. 8. Pressure distribution under a cone and pressure distribution under the corresponding wedge

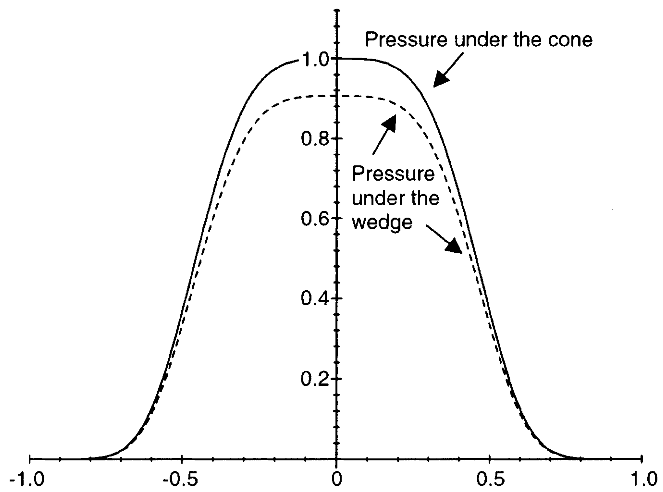

Fig. 9. Pressure distribution under a cone and pressure distribution under the corresponding wedge for Gaussian pressure and no dip in the middle

The pressure under the middle of the cone vanishes, whereas the pressure distribution of the wedge retains a relative minimum only where the pressure is about 80 percent the pressure of the maximum. The area under the curve is of course larger because the weight of a wedge of length r is larger than the weight of a cone with the same cross section.

4.3
Gaussian pressure distribution

If one chooses the pressure distribution of the cone as Gaussian, so that there is a maximum in the pressure distribution of the heap, e.g.

$$P_3^{\text{cone}}(x, y) = \mathrm{e}^{-a(\,y^2 - x^2)}, \tag{23}$$

the resulting pressure distribution is an error function (see also Fig. 8):

$$P_3^{\text{wedge}}(y) = \frac{\sqrt{\pi}\,\mathrm{erf}(a\sqrt{1-y^2})}{a\mathrm{e}^{a^2 y^2}}. \tag{24}$$

The profile of the pressure curve has not changed much, but the pressure under the wedge is a bit flattened out compared to the cone, as can be seen in Fig. 8.

4.4
Flat pressure in the middle

If one chooses the pressure distribution as flat in the middle, e.g.

$$P_4^{\text{cone}}(x, y) = \mathrm{e}^{-ay^4}, \tag{25}$$

which models quite well the pressure in the middle of some of the heaps which were used by Liffman et al in [3, 4], then the resulting pressure distribution for the wedge becomes

$$P_4^{\text{wedge}}(y) = \mathrm{e}^{-ay^4} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \mathrm{e}^{-ax^4}\,\mathrm{d}x. \tag{26}$$

Hence the pressure profile for a flat pressure in the middle is practically the same for a cone and for a wedge, as can be seen from Fig. 9 where the curves are plotted with $a = 5.0625$.

4.5
Minimum in the middle using Gaussian distributions

If one constructs a wedge with a pressure minimum, which can be modeled by superimposing Gaussians such as

$$P_5^{\text{cone}}(x, y) = \mathrm{e}^{-a^2(y^2 - x^2)} - c\mathrm{e}^{-b^2(y^2 - x^2)}, \tag{27}$$

the resulting pressure for the cone is

$$P_5^{\text{wedge}}(y) = \frac{\sqrt{\pi}\,\mathrm{erf}(a\sqrt{1-y^2})}{a\mathrm{e}^{a^2 y^2}} - \frac{\sqrt{\pi}\,\mathrm{erf}(b\sqrt{1-y^2})}{b\mathrm{e}^{b^2 y^2}} \tag{28}$$

The pressure distribution for $a = 2, b = 8, c = 0.5$ is plotted in Fig. 10. Parabola-shaped pressure distributions

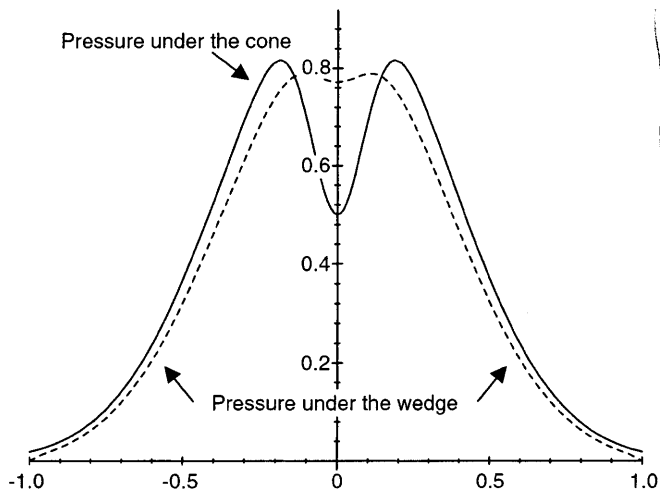

Fig. 10. Pressure distribution under a cone and pressure distribution under the corresponding wedge for Gaussian pressure and no dip in the middle

Table 1. Relative weights of the heap for the experimental and predicted heaps in Fig. 11.

Curve	Experimental	Prediction, Volume Normalization	Prediction, Height Normalization
Weight	1.0	1.15	1.21

behave similarly in the middle. Only the distribution near the border is slightly differerent for wedges and cones.

5 Comparisons with experiments

The above arguments are only qualitative. The validity of the reasoning can only be judged by a comparison with experiments. One cannot expect quantitative agreement, but one should expect qualitative agreement. The only experiment which allows a verification of the above predictions is the paper by Hummel in ref. [9]. The results have been questioned in a theoretical paper by [21], but other studies like [16] consider the experiment as reliable. There are experiments in which particles are dropped from some height into a silo, and the density in the middle of the silo, where a kind of "funnel" of high density is formed, is larger than near the walls [22]. The situation may be the same here. However, the filling process was a bit different, because in [9], it was explicitly stated that the dropping height was held constant to avoid exactly this kind of effect.

We reproduce the original data for the cone "A" in [9] with 18 inches height, the data for the wedge with 17 inches height and our predictions in Fig. 11. We kept the units from Hummel for the annotation for the y-axis, so that the pressure is measured in "inches of sand".

We used two different normalizations: One normalization was chosen to give the same height for the wedge as for the cone. To monitor the effect that the wedge from linear superposition has not straight slopes, we also normalized one pressure curve according to Eq. (10) in such a way that the volume was conserved to allow for the difference in the profiles of the heap. We used linear interpolation for the three dimensional numeric integration of the measurement data, but cubic interpolation gave practically the same shape.

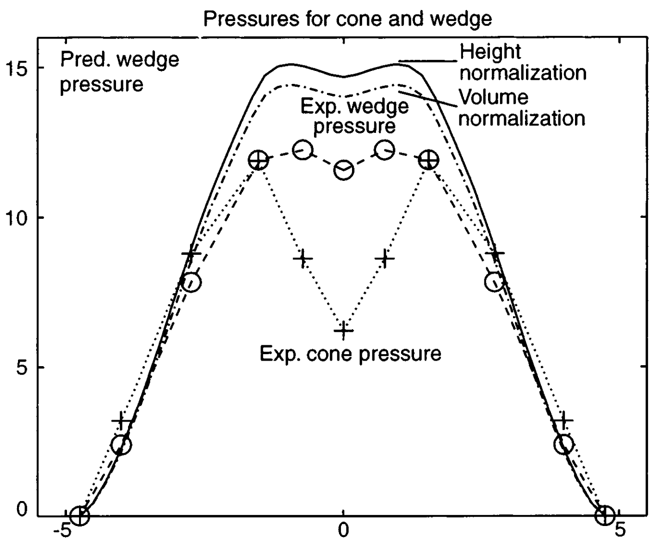

Fig. 11. Pressure distribution under a cone and pressure distribution under the corresponding wedge for a pressure distribution with the minimum modeled by two Gaussians

The qualitative correspondence is remarkable for such a handwaving theory, the small dip for the wedge is reproduced reasonably well from the cone data. The total magnitude deviates strongly. The relative weight which can be computed for the heap can be seen in Table 1.

6 Conclusions

We have attempted to compute the pressure under granular wedges by the linear superposition of granular cones. This was not done to claim that the pressures or stresses in granular aggregates can actually be described by linear superposition, but in an attempt to keep the necessary mathematical operations as simple as possible and to obtain parameter–free predictions.

Even in the most simple case of linear superposition of pressures, the pressure distribution of cones seems to differ markedly from that of wedges.

It has been concluded from experiments that the construction history is not important for the pressure distribution in wedges: "The result for the three types of loading sequence were, within experimental accuracy, identical." ([14], cited after [23]). This can be easily explained by the averaging effect along one axis of the heap. Therefore, one cannot conjecture that the construction history is unimportant for cones as well. The effect of the construction history in 2D heaps has recently been investigated in [8]. The static indetermancy of frictional contacts may influence the macroscopic observables of cones, whereas this effect will probably not be noticeable for wedges.

Two-dimensional geometries seem to be more similar to cone geometries, not to wedge geometries because there is no averaging along one axis. Therefore, the variety of pressure distributions is larger, ranging from dip's to practically flat distributions [8]. Experimentally, two-dimensional simulations describe the cut through an aggregate of two-dimensional rods, which is experimentally implemented by the so-called "Schneebeli-material" as used in [24]. The absence of the pressure dip under wedge experiments and its existence under cone experiments can therefore be explained by purely geometric reasons.

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