ROTATION AND REPTATION

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In order to understand the peculiar behavior of granular matter, it is often elucidating to observe the physics of only a few grains. We present two setups which fall into this class: The motion of a single particle in a rotating drum, and the collective behavior of a few particles under the influence of a swirling motion.

1. Rotation

1.1. INTRODUCTION

Friction is important in understanding the behavior of granular matter. Motivated by discrepancies between numerical simulation [1] and experiment [2], we have investigated the motion of a single sliding particle in a rotating drum. A fascinating feature of this setup from the point of view of the theorist is that small changes of the friction law lead to different behaviors of the particle.

Figure 1. Experimental setup and theoretical simplification (left) of the drum

The experimental setup assures that particle movement is restricted a to one-dimensional trajectory, hence the position can be described by $\varphi(t)$. Let the angular velocity of the drum be ω, its radius R. Since the friction force F_{fric} is proportional to the normal force F_{norm}, the differential equation for this problem takes the following form, if we assume the friction coefficient to be a function of the relative velocity $v_{rel} = R \cdot (\omega - \dot{\varphi}) = R\dot{\varphi}_{rel}(t)$:

$$\mathrm{R}\ddot{\varphi}(\mathrm{t}) = -\mathrm{g}\sin\varphi(\mathrm{t}) + \mu(\dot{\varphi}_{rel}(\mathrm{t})) \cdot \Big\{\mathrm{g}\cos\varphi(\mathrm{t}) + \mathrm{R}\dot{\varphi}^2(\mathrm{t})\Big\}. \tag{1}$$

For *constant* μ ($\mu = \mu_0$), this equation can be integrated once, which yields

$$\dot{\varphi}^2 = -2\frac{g}{R}\frac{1}{1+4\mu_0^2} \cdot \Big\{(2\mu_0^2 - 1)\cos\varphi - 3\mu_0 \sin\varphi\Big\} + 2e^{2\mu_0\varphi} \cdot \mathrm{c}. \tag{2}$$

For appropriate values of c, this describes periodic motion. Therefor, assuming that the presence of the perturbation will change the constant of integration c into a slowly varying function of time: $c = c_0(\tau) + \varepsilon c_1(t,\tau) + \cdots$ where $\tau = \varepsilon t$ we are led in a natural way to the method of averaging [3, 4, 5]. Differentiating c once yields $\dot{c} = \varepsilon(c_{0\tau}(\tau) + c_{1t}(t,\tau)) + \mathcal{O}(\varepsilon^2)$.

We can eliminate c_1 via integrating over one period and using the time derivative of equation (2), we obtain

$$\varepsilon c_{0\tau} = -\frac{2}{\mathrm{T}_p}\int_{\varphi^{min}}^{\varphi^{max}} \mathrm{d}\varphi \; \dot{\varphi} e^{-2\mu_0\varphi}\dot{\varphi} \cdot \varepsilon \left[(\mu(\dot{\varphi}) - \mu_0) \cdot \left(\frac{\mathrm{g}}{\mathrm{R}}\cos\varphi + \dot{\varphi}^2\right)\right] \tag{3}$$

where T_p is the periodicity. Of course, T_p itself depends on c_0 via equation (2). If equation (3) happens to have a fixed point c_0^*, there is a periodic solution of the *perturbed* equation, the periodicity of which is given by $T_p(c_0^*)$.

Figure 2. Different types of fixed points

Examining figure 2, we can see three different types of fixed points. In case (a), we have a stable orbit, case (b) is unstable. In case (c), where $c_{0\tau} = 0$ on a whole interval, we have a continuum of marginally stable orbits.

1.2. FRICTION LAWS

1.2.1. *Coulomb's law*

In this case, the friction coefficient depends only on the direction of the velocity vector (Coulomb's law).

$$\mu(\dot{\varphi}_{rel}) = \begin{cases} \mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} \geq 0 \\ -\mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} < 0 \end{cases} .$$

We observe two different kinds of trajectories: For small starting values $\varphi(0)$ and $\dot{\varphi}(0) = 0$, the particle velocity will be small and $\dot{\varphi}_{rel}$ is always > 0. Hence, we have the unperturbed solution, the trajectory is marginally stable and $c_{0\tau} \equiv 0$.

For larger values of $\varphi(0)$ and $\dot{\varphi}(0) = 0$, starting conditions lie outside the periodic orbit and $\dot{\varphi}_{rel}$ can become < 0. As a result, the particle dissipates energy, the trajectories approach the periodic orbit.

Figure 3. (a) shows a phase plot $\omega = 1$, $\mu = 0.4$ R $= 1$ and $\varphi(0) = 0.5, 0.7, 1.0, 1.2$; (b) shows $c_{0\tau}$ as a function of c_0[equation (3)] for Coulomb's law. The parameters are the same as in (a).

1.2.2. *Friction law as suggested by Rabinowicz*

E. Rabinowicz.[6] proposed a friction law which gives a velocity dependence of $\mu \propto v^{-0.1}$.

$$\mu(\dot{\varphi}_{rel}) = \begin{cases} \mu_{kin}\dot{\varphi}_{rel}^{-0.1} & \text{if} \quad \dot{\varphi}_{rel} \geq \dot{\varphi}_0 \\ -\mu_{kin}\dot{\varphi}_{rel}^{-0.1} & \text{if} \quad \dot{\varphi}_{rel} < -\dot{\varphi}_0 \end{cases}$$

Figure 4. (a) shows a phase plot $\omega = 1$, $\mu = 0.4$ R $= 1$ and $\varphi(0) = 0.5, 0.7, 1.0, 1.2$ (b) shows a typical plot for equation (3) if we have the friction law suggested by Rabinowicz. The parameters are the same as in (a).

For $\varphi(0) < \varphi(0)_{\text{periodic}}$ (and $\dot{\varphi}(0) = 0$), the particle gains energy and approaches the periodic state. Then there is the periodic orbit itself and for $\varphi(0) > \varphi(0)_{\text{periodic}}$, the trajectories approach the stable orbit due to dissipation of energy. Details about yet another friction law are given in [5].

1.3. DISCUSSION

Comparing this behavior with experimental results,[2] we obtain good agreement using Rabinowicz's friction law.

Coulomb's law with or without static friction [5] is not able to reproduce the typical behavior of the experiment. The reason is that with Coulomb's law, the fixed point of (1), given by $\tan(\varphi) = \mu_0$, is elliptic, hence structurally unstable and destructible by arbitrarily small perturbations.

2. Reptation

Swirling a single layer of spheres in a horizontally oriented circular container, we observe two different dynamical modes: For small numbers of spheres the cluster follows the direction of the orbital motion whereas the sense of rotation changes when the number of spheres N exceeds a critical value [7]. The first mode is called rotation and the second reptation. In addition to former findings we present experiments for the case where the ratio of particle diameter d to diameter of the circular container D is small. Recent numerical simulations suggest that in this case the rotation mode is suppressed and only reptation occurs [8].

2.1. EXPERIMENTAL SETUP

To investigate the behavior of granular material under a swirling motion we use an adjustable reciprocating orbital shaker (Thermolyne AROS 160) as shown in [7]. Every point of the shaking table performs the same orbital movement, there is no center of rotation. The driving frequency f_d of the shaker is fixed to 1.5 Hz. By a mechanical adjustment we can examine four different driving amplitudes A_d of the orbital motion: 6.35, 9.53, 12.70, and 15.88 mm. A Petri dish with an inner diameter of 176 mm is fixed on the swirling table. As granular material we use glass marbles with a mean diameter of 15.52 mm. The material density is given by 2.4 g/cm^3. The advantage of these marbles is that although they are of the same material the inside contains spots of different colors. Thus the path of a single sphere can be easily visualized while it is ensured that the colliding surfaces have the same material properties.

We start with the closest possible packing density and measure the time T_s a single sphere needs to circumnavigate the container. We focus on a sphere which is close to the boundary of the Petri dish, i.e. a particle in the outer ring of the cluster is used as a tracer to indicate the dynamics of the granular material. T_s is measured ten times and averaged. Next, one particle is removed and after a waiting time of 3 min we again determine the period of revolution.

2.2. EXPERIMENTAL RESULTS

The influence of the packing density p on the normalized frequency of rotation f_n for different driving amplitudes is seen in Fig. 5. f_n is the ratio of f_s and f_d, where $f_s = 1/T_s$. As packing density we define a two dimensional solid fraction: $p = Nd^2/D^2$. While the packing density is decreased by steps of the amount of d^2/D^2 we study the response of the tracer sphere. It is observed that at high packing densities the outer sphere does not exit the outer layer of the cluster during one revolution of the cluster. But, at a certain critical packing density this behavior changes because the mobility of the sphere increases dramatically. Therefore it is likely that the tracer particle travels to the second inner ring of the cluster. At this stage it becomes questionable to follow the path of a single sphere in order to obtain information of the global dynamics of the whole cluster. Thus, no more measurements are performed below this critical packing density. Nevertheless it is found that above this threshold only the reptation mode is observed, which is expressed by a negative normalized frequency of rotation because the rotation of the cluster is direct opposite to the swirling motion.

For small driving amplitudes (A_d = 6.35 and 9.53 mm) we obtain a parabolic response behavior in Fig. 5. This means that as the packing density is decreased the angular velocity of our cluster first increases and then decreases again. For $A_d = 12.70$ mm we observe that there is a deformation in the parabolic shape. This behavior is even enhanced for the largest driving amplitude ($A_d = 15.88$ mm). In this case the data points are w-shaped.

2.3. DISCUSSION

The most interesting feature of Fig. 5 is the w-shaped behavior of the normalized frequency of rotation for $A_d = 15.88$ mm. This means that in a certain range the same rotational speed of the cluster is found for four different numbers of spheres. To explain this we speculate that for a certain amplitude of driving f_n is determined rather by the size of the cluster than the packing density. Since we are in a regime where sheared granular material expands its volume, which is known as Reynolds dilatancy [9], it is likely that different numbers of spheres could result in the same cluster size and thus give the same frequency of rotation. In our case it seems that in a certain range of the packing density it makes no difference whether the cluster is densely or loosely packed. We conclude that swirling granular material could be used to determine the range where Reynolds dilatancy

Figure 5. The frequency of rotation of a tracer sphere in the outer layer of the reptating cluster is shown in dependence on the packing density. The frequency of rotation is given in units of the driving frequency of the orbital shaker. The measurements represent runs for four different driving amplitudes. The curves are obtained by polynomial fits of second ($A_d = 6.35$ and 9.53 mm), fourth ($A_d = 12.70$ mm), and sixth order ($A_d = 15.88$ mm) and should serve as a guide to the eye.

occurs: The limits are given by the packing densities of the two corresponding local maxima in the frequency of rotation. To support this idea runs with even larger container sizes and/or smaller particle sizes have to be performed where local density measurements should uncover the different packing configurations.

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